Entanglement, von Neumann Entropy and the GNS construction

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Outline

Introduction

- In studies of foundations of quantum theory, it is of interest to study mixed states and their origins.
- Focus has been on separable states and entropy created by partial tracing.
- But this method is not so good for identical particles as we will show.
- A much more universal construction is based on restrictions of states to subalgebras and the GNS construction.
- This talk will explain this approach

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WHAT THEY DO

Separable state

Consider a bipartite system with Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$.
A vector state

$$|\psi\rangle = \sum_{i,j} \psi_{ij} |i\rangle \otimes |j\rangle$$

is said to be *separable* if it can be brought to the form

$$|\psi\rangle = |v\rangle \otimes |w\rangle.$$

Otherwise, it is said to be *entangled*.

Singular value decomposition (SVD)

A : $m \times n$ complex matrix. A can always be written in the form

$$A = UDV^\dagger$$

- U : $m \times m$, unitary (columns of U are eigenvectors of $AA^\dagger$).
- D : $m \times n$, diagonal, positive (eigenvalues of $A^\dagger A$.)
- V : $n \times n$ unitary (columns of V are eigenvectors of $A^\dagger A$).

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Schmidt decomposition

$$|\psi\rangle = \sum_{i,j} A_{ij}|i\rangle \otimes |j\rangle = \sum_{i,j} (UDV^\dagger)_{ij}|i\rangle \otimes |j\rangle$$

$$= \sum_{i,j,k,l} U_{ik} \underbrace{D_{kl}}_{=\lambda_k \delta_{kl}} V^\dagger_{lj}|i\rangle \otimes |j\rangle = \sum_k \lambda_k \left(\sum_i U_{ik}|i\rangle\right) \otimes \left(\sum_j \bar{V}_{jk}|j\rangle\right)$$

$$= \sum_k \lambda_k |k\rangle_A \otimes |k\rangle_B$$

Schmidt number

The Schmidt number is the number of nonzero λ_k's.

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The Schmidt number is the number of nonzero λ_k's.

- $|\psi\rangle$ separable precisely when Schmidt number = 1.
- Reduced density matrix: $\rho_A = \mathrm{Tr}_B \rho$.
- von Neumann entropy: $S(\rho) = -\mathrm{Tr}\rho \log \rho$. We have $S(\rho_A) = S(\rho_B)$.
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WHAT WE DO

Our main motivation

There are certain situations where the use of partial trace may not be the "best thing to do". An example of this is provided by the study of entanglement for systems of indistinguishable particles, where the notion of separability is more subtle.

Idea

Partial trace = Restriction

Thus consider two distinguishable particles A and B in a pure state $|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B$. Partial trace means restricting its density matrix to observables of subsystem A . They are of the form $K_A \otimes \mathbb{1}_B$. But for identical fermions....

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Identical Fermions

For identical fermions a two particle state is a linear combination of vector states of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\phi\rangle \otimes |\chi\rangle - |\chi\rangle \otimes |\phi\rangle)$$

and observables are all symmetric combinations $K \otimes L + L \otimes K$. Partial tracing has no physical meaning. How do we study the mixture created by observing only the single particle observables? $K \otimes \mathbb{1} + \mathbb{1} \otimes K$ or perhaps $L \otimes L$? We turn now to this problem.

C*-algebras

Observables in quantum field theory come from C^*-algebras. All finite-dimensional matrix algebras are C^*.

Representations of C^*-algebras

Given a state or density matrix on such an algebra, there is a way to recover the Hilbert space due to Gelfan'd, Naimark and Segal. We will explain it below. It is used widely in

- Quantum field theory.
- Statistical physics.
- Noncommutative geometry.

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The GNS construction

- Let $\mathcal{A}$ be the C^*-algebra of observables and $\omega : \mathcal{A} \to \mathbb{C}$ a state. That is $\omega(\alpha)$ is a complex number, $\omega(\mathbb{1}_{\mathcal{A}}) = 1$.
- Regard $\mathcal{A}$ as a vector space: $\alpha \to |\alpha\rangle$.
- Introduce a scalar product: $\langle\alpha|\beta\rangle = \omega(\alpha^*\beta)$
- This space can have a subspace $\mathcal{N}$ of vectors of 0 norm: $\mathcal{N} = \{\alpha \in \mathcal{A} \mid \langle\alpha|\alpha\rangle = 0\}$.
- The Hilbert space is: $\mathcal{H}_\omega = \mathcal{A}/\mathcal{N}$. An element of this space is the equivalence class $[\alpha] = \alpha + \mathcal{N}$ for α in $\mathcal{A}$.
- The representation L of $\mathcal{A}$ on this space is given by $L(\alpha)|[\beta]\rangle = |[\alpha\beta]\rangle$

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- Fact: $\mathcal{H}_\omega$ irreducible precisely when ω pure.
- That is, the restriction of ω to L is $\rho_\omega = |[\mathbb{1}]\rangle\langle[\mathbb{1}]|$, so that it is of rank 1.
- $\mathcal{H}_\omega = \oplus_j \mathcal{H}_j$, where each $\mathcal{H}_j$ carries an irreducible representation.
- If P_j's are projectors from $\mathcal{H}_\omega$ to $\mathcal{H}_j$, set $|[\mathbb{1}_j]\rangle = \frac{P_j|[\mathbb{1}]\rangle}{\|P_j|[\mathbb{1}]\rangle\|}$.
- Define $\mu_j = \|P_j|[\mathbb{1}]\rangle\|$. Then , $\sum_j \mu_j^2 = 1$. Finally,

$\rho_\omega = \sum_j \mu_j^2 |[\mathbb{1}_j]\rangle\langle[\mathbb{1}_j]|$. This state is mixed if its rank exceeds 1. It has entropy

$$S = -\sum_j \mu_j^2 \log \mu_j^2 .$$

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EXAMPLE

- Single-particle space: $\mathcal{H} \simeq \mathbb{C}^d$.
- Symmetry group: $U(d)$.
- Algebra of observables: $\mathcal{A}$, given by a $*$-representation of $\mathbb{C}U(d)$ for the group algebra:

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Consider now a fermion system whose single-particle space is given by $\mathcal{H} = \mathbb{C}^d$:

- Many particle space $\mathcal{F}$ is the “Fock” space:

$$\mathcal{F} = \bigoplus_{k=0}^{d} \mathcal{H}^{(k)}, \text{ where } \mathcal{H}^{(k)} \equiv \Lambda^k \mathcal{H}.$$

- $\Lambda^0 \mathcal{H} = \mathbb{C}$, generated by the “vacuum" $|\Omega\rangle$.
- $\Lambda^1 \mathcal{H}$: 1-particle space, $\Lambda^2 \mathcal{H}$: 2-particle space, and so on..
- $\dim \Lambda^k \mathcal{H} = \binom{d}{k}$, so $\dim \mathcal{F} = 2^d$. This reflects the fact that $\mathcal{F} \cong \mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2$, an isomorphism often used in statistical physics models.

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$\{|e_1\rangle, |e_2\rangle, \dots |e_d\rangle\}$: Orthonormal basis for $\mathcal{H}$. Given $v_i \in \mathcal{H}$ $(i = 1, \dots, k)$, put

$$|v_1 \wedge \ldots \wedge v_k\rangle = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) v_1 \otimes \cdots \otimes v_k.$$

Then:

- $\{|e_{i_1} \wedge \ldots \wedge e_{i_k}\rangle\}_{1 \leq i_1 < \cdots < i_k \leq d}$: ONB for $\mathcal{H}^{(k)}$.
- A self-adjoint operator A acting on $\mathcal{H}^{(1)} \equiv \mathcal{H}$ can be made to act on $\mathcal{H}^{(k)}$ by defining $d\Gamma^k(A)$ follows:

$$d\Gamma^k(A) = A \otimes \mathbb{1}_d \cdots \otimes \mathbb{1}_d + \mathbb{1}_d \otimes A \otimes \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d + \cdots + \mathbb{1}_d \otimes \cdots \otimes \mathbb{1}_d \otimes A$$

- This operator preserves the symmetry of the states on which it acts, as well as the commutation relations of the self-adjoint operators acting on $\mathcal{H}$, namely:

$$d\Gamma^k([A, B]) = [d\Gamma^k(A), d\Gamma^k(B)]$$

$\{|e_1\rangle, |e_2\rangle, \ldots |e_d\rangle\}$: Orthonormal basis for $\mathcal{H}$. Given $v_i \in \mathcal{H}$ $(i = 1, \ldots, k)$, put

$$|v_1 \wedge \ldots \wedge v_k\rangle = \frac{1}{\sqrt{k!}} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) v_1 \otimes \cdots \otimes v_k .$$

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- $d\Gamma(A) = \sum_k d\Gamma^k(A)$ acts on the whole Fock space $\mathcal{F}$ (it is the "second quantized" form of A).
- At the group level, we may consider exponentials of such operators, of the form $e^{id\Gamma(A)}$. These are unitary operators acting on $\mathcal{F}$. Let $U = e^{iA}$ be a unitary operator acting on $\mathcal{H}$. The global version of $d\Gamma$ is given by:

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Following the previous remarks, we see that operators of the form

$$\widehat{\alpha}^k = \int_{U(d)} d\mu(g)\alpha(g)U^{(1)}(g) \otimes \cdots \otimes U^{(1)}(g) \quad (k\text{-fold product}),$$

act properly on $\mathcal{H}^{(k)}$. All of this can be conveniently expressed in terms of a *coproduct*. In fact, an approach based on Hopf algebras can automatically include braid-group statistics. Here, the construction of the observable algebra corresponds to the following simple choice for the coproduct Δ:

$$\Delta(g) = g \otimes g, \quad g \in U(d), \tag{1}$$

linearly extended to all of $\mathbb{C}U(d)$. This choice fixes the form of $\widehat{\alpha}^k$.

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AN EXAMPLE

- Consider the 2-fermion space $\mathcal{H}^{(2)}$ for the case $d = 3$.
- Let the action of $U(3)$ on $\mathcal{H} = \mathbb{C}^3$ be given by the defining representation (hence $U^{(1)}(g) = g$).
- We have $U^{(1)}(g)|e_i\rangle = \sum_{j=1}^{3} D(g)_{ij}|e_j\rangle$, for a fixed ONB $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$.
- The action of $\mathbb{C}U(3)$ is then given by the 3-dimensional conjugate representation ($3 \otimes 3 = 6 \oplus \bar{3}$).
- $\bar{3}$ is the antisymmetric $3 \otimes_A 3$.
- The basis vectors of $\bar{3}$ are $|e^k\rangle := \varepsilon^{ijk}|e_i \wedge e_j\rangle$ $(k = 1, 2, 3)$.

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Action of $\mathbb{C}U(3)$ on $\mathcal{H}^{(1)}$: $\widehat{\alpha} = \int_{U(3)} d\mu(g)\alpha(g)D^{(3)}(g)$.
Basis: the 8 Gell-Mann matrices $\{T_i\}_i$ plus $\mathbb{1}_3$:

$$T_1 = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_2 = \begin{pmatrix} 0 & -\frac{i}{2} & 0 \\ \frac{i}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, T_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$T_4 = \begin{pmatrix} 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \end{pmatrix}, T_5 = \begin{pmatrix} 0 & 0 & -\frac{i}{2} \\ 0 & 0 & 0 \\ \frac{i}{2} & 0 & 0 \end{pmatrix}, T_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix},$$

$$T_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{2} \\ 0 & \frac{i}{2} & 0 \end{pmatrix}, T_8 = \begin{pmatrix} \frac{\sqrt{3}}{6} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{6} & 0 \\ 0 & 0 & -\frac{\sqrt{3}}{3} \end{pmatrix}, \mathbb{1}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- On $\bar{3}$, they become $\widetilde{T}_j = -\overline{T}_j$ (use Δ and restrict to $\bar{3}$).
- This amounts to consider only the action of the operators

$$\widehat{\alpha}^2 = \int_{U(3)} d\mu(g)\alpha(g)D^{(3)}(g) \otimes D^{(3)}(g)$$

on the (invariant) subspace generated by the antisymmetric vectors $|e^k\rangle = \varepsilon_{ijk}|e_i \wedge e_j\rangle$ ($k = 1, 2, 3$).

Explicitly, we have, for instance:

$$\widetilde{T}_1 \cdot |e^1\rangle = (T_1 \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes T_1)|e^1\rangle = (T_1 \otimes \mathbb{1}_3 + \mathbb{1}_3 \otimes T_1)|e_2 \wedge e_3\rangle$$

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Summarizing:

The algebra of operators $\mathcal{A}$ acting on the 2-particle sector $\mathcal{H}^{(2)} = \Lambda^2\mathbb{C}^3$ is the matrix algebra generated by $\{\widetilde{T}_1, \ldots, \widetilde{T}_8, \mathbb{1}_3\}$.

If we now assume that we only have access to the observables pertaining to the states $|e_1\rangle$ and $|e_2\rangle$, then the relevant algebra of operators will be a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$, namely the one generated by $\{\widetilde{T}_1, \widetilde{T}_2, \widetilde{T}_3, \mathbb{1}_2, \mathbb{1}_3\}$.

In general we expect that a (2-particle) pure state defined in the original system that is given by a state vector

$$|\psi\rangle = \sum_{k=1}^{3} \psi_k |e^k\rangle = \psi_1|e_2 \wedge e_3\rangle + \psi_2|e_3 \wedge e_1\rangle + \psi_3|e_1 \wedge e_2\rangle$$

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In order to detect this “entanglement” we perform the GNS construction when the state $|\psi\rangle\langle\psi|$ is restricted to $\mathcal{A}_0$. Algebraically, this state is a linear functional on $\mathcal{A}$, defined as follows:

$$\begin{aligned} \omega_\psi : \mathcal{A} &\to \mathbb{C} \\ \widehat{\alpha} &\mapsto \omega_\psi(\widehat{\alpha}) := \langle\psi|\widehat{\alpha}|\psi\rangle. \end{aligned}$$

So we put $\omega_{\psi,o} = \omega_\psi \mid_{\mathcal{A}_0}$. The GNS construction furnishes a representation $L : \mathcal{A}_0 \to B(\mathcal{H}_{GNS})$ of $\mathcal{A}_0$, for a certain Hilbert space $\mathcal{H}_{GNS}$ constructed from $\mathcal{A}_0$.

In general, $\mathcal{H}_{GNS}$ will split as a sum of irreducibles of $\mathcal{A}_0$. This reducibility reflects the fact that, when restricted to $\mathcal{A}_0$, the original state ω_ψ might become *mixed*. The entropy of this state can then be computed from the decomposition of $\mathcal{H}_{GNS}$ into irreducibles.

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Vector state

$$|\psi\rangle = \cos\theta|e^1\rangle + \sin\theta|e^3\rangle$$

von Neumann entropy of $\omega_{\psi,0}$:

$$S(\theta) = \cos^2\theta \log\frac{1}{\cos^2\theta} + \sin^2\theta \log\frac{1}{\sin^2\theta}$$

Partial trace vs. GNS

In this example, partial trace always gives zero for entropy.
If $\mathbb{C}^4$ describes single-particles, there are pure states of Schmidt rank 1 with zero for GNS entropy, 1 for partial trace entropy.
The former is more reasonable, the state being least entangled.

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Dimensions

$$\mathcal{H}^{\theta}_{GNS} \cong \begin{cases} \mathbb{C}^2, & \theta = 0 \\ \mathbb{C}^3 \cong \mathbb{C}^2 \oplus \mathbb{C}, & \theta \in (0, \pi/2) \\ \mathbb{C}, & \theta = \pi/2. \end{cases}$$

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